UDC 539.3

PLANE PROBLEM OF THE THEORY OF VISCOELASTICITY WITH MOVING BOUNDARIES OF PHASE TRANSFORMATIONS

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The problem of determining the stresses in an infinite viscoelastic plate with a circular cavity surrounded by a zone of phase transformation increasing with time, is studied. The problem is reduced to a first order system of ordinary differential equations. It is well known that the time dependence of the inverse operator of the elastic problem arising, in particular, in problems in which the boundary conditions are formulated on moving boundaries (contact problems, determination of the stresses in burning cylinders, etc.) excludes, as a rule, the possibility of using the Volterra principle [1-3]. The present paper is concerned with the problem of determining the stresses in an infinite viscoelastic plate with a circular cavity of radius a, around which a region of phase transformation increasing with time is formed. The physical characteristics of the medium such as density, Young's modulus, viscosity, etc., vary at the phase boundary $S(t)$ discontinuously. Paper [4], used the framework of the theory of elasticity to investigate the stresses caused by changes in density during a phase transformation in a semiplane under the action of a thermal stamp, with the interphase boundary remaining stationary.

We shall assume that the temperature field is axisymmetric, from which it follows that the phase boundary is circular. The law of its motion can be found by solving the corresponding Stefan problem [5].

We write the defining stress-strain relation in the form

$$\dot{\varepsilon}_{x,y} = \frac{1+\nu}{\mathrm{E}}\{\sigma_{x,y} - \nu(\sigma_x + \sigma_y)\}, \quad \dot{\varepsilon}_{xy} = \frac{1+\nu}{\mathrm{E}}\sigma_{xy}$$

$$\frac{1}{\mathrm{E}} = \frac{1}{E}\left\{1 + \sum_k a_k \frac{d^k}{dt^k}\right\}$$

Using the Airy function and the equation of compatibility of deformations, we can obtain the analog of the Kolosov-Muskhelishvili [6] formulas

$$u_1^{\boldsymbol{\cdot}} + iu_2^{\boldsymbol{\cdot}} = \frac{1+\nu}{E}\{\varkappa\varphi(z,t) - z\overline{\varphi'(z,t)} - \overline{\psi(z,t)}\} \tag{1}$$

$$\sigma_{11} + \sigma_{22} = 2\operatorname{Re}\varphi'(z,t)$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\{\bar{z}\varphi''(z,t) + \psi'(z,t)\}$$

Let the region $a \leqslant r \leqslant S(t)$ be occupied by phase 1, and the region $\infty > r > S(t)$ by phase 2. We supplement equations (1) with the boundary conditions

$$\sigma_{11}^{(2)} = -p_1, \quad \sigma_{22}^{(2)} = -p_2, \quad r = \infty \tag{2}$$
$$\sigma_r^{(1)} - i\sigma_{\theta r}^{(1)} = \sigma_r^{(2)} - i\sigma_{r\theta}^{(2)}, \quad r = S(t)$$
$$u_r^{\cdot(1)} - iu_\theta^{\cdot(1)} = u_r^{\cdot(2)} - iu_\theta^{\cdot(2)}, \quad r = S(t)$$
$$\sigma_r^{(1)} - i\sigma_\theta^{(1)} = 0, \quad r = a$$

The above conditions may correspond, e.g. to the problem of stability of an unsupported deep mine in frozen rock. Here we neglect the density changes, and the influence of the phase transformation of the interstitial water is reflected in such characteristics as Young's modulus, viscosity, etc.. The upper index accompanying the variables indicates the zone (one of two) to which the quantity in question refers.

Let us write the unknown functions $\varphi(z, t)$ and $\psi(z, t)$ for each zone in terms of the following power series [7]:

$$\begin{aligned} &\varphi_1(z, t) = \{a_3(t)z^3 + a_1(t)z + a_{-1}(t)z^{-1}\} \\ &\psi_1(z, t) = \{b_1(t)z + b_{-1}(t)z^{-1} + b_{-3}(t)z^{-3}\} \quad \text{when } a \leqslant r \leqslant S(t) \end{aligned} \tag{3}$$

$$\left.\begin{aligned} &\varphi_2(z, t) = \Gamma\left\{z + \frac{\alpha_{-1}(t)}{z}\right\} \\ &\psi_2(z, t) = \Gamma'\left\{z + \frac{\beta_{-1}(t)}{z} + \frac{\beta_{-3}(t)}{z^3}\right\} \end{aligned}\right\} \text{when } S(t) \leqslant r < \infty$$

The constants Γ and Γ' are chosen in such a manner, that the boundary conditions at infinity are satisfied. Substituting the expansions (3) into the last three boundary conditions of (2) and comparing the coefficients of like powers of $e^{i\theta}$, we obtain the following set of equations determining the unknown functions appearing in (3):

$$2a_1(t) + \frac{b_{-1}(t)}{a^2} = 0, \quad a_3(t)\,a^4 - a_{-1}(t)\,a^2 + b_{-3}(t) = 0, \tag{4}$$
$$3a_3(t)\,a^4 + a_{-1}(t) + b_1(t)\,a^2 = 0$$
$$2\Gamma + \frac{\Gamma'\beta_{-1}(t)}{S^2(t)} = 2a_1(t) + \frac{b_{-1}(t)}{S^2(t)}$$
$$-\frac{\Gamma\alpha_{-1}(t)}{S^2(t)} + \frac{\Gamma'\beta_{-3}(t)}{S^4(t)} = a_3(t)\,S^2(t) - \frac{a_{-1}(t)}{S^2(t)} + \frac{b_{-3}(t)}{S^4(t)}$$
$$\frac{\Gamma\alpha_{-1}(t)}{S^2(t)} + \Gamma = 3a_3(t)\,S^2(t) + \frac{a_{-1}(t)}{S^2(t)} + b_1(t)$$
$$\frac{1+\nu_2}{E_2}\left\{(\varkappa_2 - 1)\,S(t)\,\Gamma - \frac{\Gamma'\beta_{-1}(\tau)}{S(t)}\right\} = \frac{1+\nu_1}{E_1}\left\{(\varkappa_2 - 1)\,a_1(\tau)\,S(t) - \frac{b_{-1}(\tau)}{S(t)}\right\}$$
$$\frac{1+\nu_2}{E_2}\left\{\varkappa_2\frac{\Gamma\alpha_{-1}(\tau)}{S(t)} - \Gamma' S(t)\right\} = \frac{1+\nu_1}{E_1}\left\{\varkappa_1\frac{a_{-1}(\tau)}{S(t)} - 3a_3(\tau)\,S^3(t) - b_1(\tau)\,S(t)\right\}$$
$$\frac{1+\nu_2}{E_2}\left\{\frac{\alpha_{-1}(\tau)\,\Gamma}{S(t)} - \frac{\Gamma'\beta_{-3}(\tau)}{S^3(t)}\right\} = \frac{1+\nu_1}{E_1}\left\{\varkappa_1 a_3(\tau)\,S^3(t) + \frac{a_{-1}(\tau)}{S(t)} - \frac{b_{-3}(\tau)}{S^3(t)}\right\}$$

In contrast with the corresponding elastic problem, the third condition of (2) yields three differential equations (last three equations of (4)). The argument τ shows that the differential operators act only on the function containing this argument. The fact that the differential equations in question contain the time—dependent cofactors accompanying the unknown coefficients of the expansions, indicates that in the problem under consideration we cannot use just any viscoelastic analogy, and the system (4) must be solved indirectly, e.g. using the method of a small parameter, if the law governing the motion of the boundary has the form $S(t) = S_0 + \varepsilon h(t), \varepsilon \ll 1$.

As an example, we shall determine the coefficient $\beta_{-1}(t)$, and we shall seek it in the form of the following series:

$$\beta_{-1}(t) = \sum_{k=0}^{\infty} \varepsilon^{(k)} \beta_{-1}^{(k)}(t)$$

The first, fourth and seventh equation of (4) can be reduced to a single differential equation in $\beta_{-1}(t)$ ·

$$\frac{1+\nu_2}{E_2}\left\{(\varkappa_2 - 1) S(t) \Gamma - \frac{\Gamma' \beta_{-1}(\tau)}{S(t)}\right\} = \frac{1+\nu_1}{E_1}\left\{\frac{(\varkappa_1 - 1) S(t)}{2} + \frac{a^2}{S(t)}\right\}\left\{\frac{2\Gamma S^2(\tau) + \Gamma' \beta_{-1}(\tau)}{S^2(\tau) - a^2}\right\} \tag{5}$$

In the zero approximation $S(t)$ and $S(\tau)$ in (5) are replaced by S_0, and this corresponds to the viscoelastic problem with a stationary boundary, a solution of which can be obtained with the help of the principle of correspondence.

The formulation given above can be generalized to the case of discontinuous density change, by replacing the third condition in (2) by

$$\dot{u}_r^{(1)} - \dot{u}_r^{(2)} \big|_{r=S(t)} = k \dot{S}(t), \quad k = \frac{\rho_2 - \rho_1}{\rho_1}$$

where k is the relative volume change and ρ_1, ρ_2 are the phase densities.

We note that if the defining stress - deformation equations have the form

$$\sigma = f(\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon}, \ldots) \tag{6}$$

then the construction of the Kolosov - Muskhelishvili formulas in terms of the Airy function for the region $a \leqslant r \leqslant S(t)$ may encounter some difficulties when $S(t)$ increases monotonously. Another method of constructing the Kolosov - Muskhelishvili formulas is given in the literature. It consists of direct transformation of the equilibrium equations written in terms of the displacements [8]. This method makes possible the direct utilization of Eqs. (6) but yields, for the displacement vector, an expression of the form

$$2G(u_1 + iu_2) = \varkappa \varphi(z, t) - z\overline{\varphi'(z, t)} - \overline{\psi(z, t)}$$

To determine the displacement from the above equation we must again perform integration with respect to time. At the same time, use of (1) instead of (6) leads to relatively simple expressions.

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